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Kaluza–Klein Theory and Dirac Equation in Higher Dimensional Riemann–Cartan Space

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The higher dimensional Kaluza–Klein theory in Riemann–Cartan space is discussed. To clarify its implications, we investigate the simplest five-dimensional case of the theory in detail. The Einstein-like, Maxwell, and Dirac equations in four-dimensional space-time are obtained by reducing the corresponding five-dimensional field equations. The effect of spin–spin interaction induced by torsion is revealed by analyzing the Dirac equation in this case.

1. INTRODUCTION

It is well known that all kinds of elementary particles can be classified by means of irreducible representations of the Poincaré group and can be labeled by mass and spin. Mass and spin are elementary notions, each with an analogous standing not reducible to the other. General relativity based on the torsion-free Riemann space-time V_4 couples the energy-momentum tensor of matter to the Riemann space-time curvature. On the other hand, following Einstein, Hayashi and Shirafuji (1979) built up a teleparallel theory of gravity in curvature-free Weitzenböck space-time A_4, in which the energy-momentum tensor of the matter couples to the torsion tensor of the space-time and the spin distribution of the matter is only reflected in the antisymmetry of the energy-momentum tensor. A more reasonable model should consider both the mass and spin of matter as sources of the curvature and torsion of space-time, respectively, and formulate a theory of gravity in Riemann–Cartan space-time U_4, which is the so-called Einstein–Cartan theory (EC theory) (Hehl *et al.*, 1976).

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The Kaluza–Klein theory (KK theory) (Toms, 1984) was first put forward by Kaluza (1921) and Klein (1926) in the early twenties. Later, it was applied to the case of higher dimensional Riemann space. The higher dimensional KK theory in Weitzenböck space has also been formulated (You-lin, 1993). Next we establish the higher dimensional KK theory in Riemann–Cartan space.

In this paper, we first give the general gravitational field equation and Dirac equation of the matter field in higher dimensional EC theory. Then, as an example, we deduce the Einstein-like equation, Maxwell equation, and Dirac equation in four-dimensional (4D) space with a given metric by reducing the corresponding five-dimensional (5D) field equations. Finally, we discuss the gravitational spin–spin interaction induced by torsion.

2. ACTION

First, we give our conventions. Objects or quantities with (without) a caret refer to those of higher dimensional space (4D space-time) and objects or quantities with a tilde are those of the Riemann space so as to distinguish them from those of the Riemann–Cartan space; Greek letters μ, ν, ... (Latin letters A, B, ...; M, N, ...) are used for coordinate basis indices (horizontal lift basis indices; orthonormal basis indices).

In order to set up theory of gravity in n-dimensional (nD) Riemann–Cartan space U_n, the nD action of the theory should be given. For this, we first define the higher dimensional torsion tensor in the coordinate basis as

$$\hat{Q}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} \equiv \hat{\Gamma}^{\hat{\lambda}}_{[\hat{\mu}\hat{\nu}]}, \qquad \hat{\mu}, \hat{\nu}, \ldots = (0, 1, 2, 3, 5, \ldots, n) \tag{2.1}$$

where $\hat{\Gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}}$ is the affine connection in Riemann–Cartan space. If the metricity condition $\hat{\nabla}_{\hat{\mu}}\hat{g}_{\hat{\lambda}\hat{\nu}} = 0$ is assumed, we have

$$\hat{\Gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}} = \hat{\tilde{\Gamma}}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}} - \hat{K}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} \tag{2.2}$$

where $\hat{\tilde{\Gamma}}^{\hat{\lambda}}_{\hat{\mu}\hat{\nu}}$ represents the corresponding Riemann connection and $\hat{K}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}}$ stands for the contortion tensor

$$\hat{K}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} = -\hat{Q}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} + \hat{Q}_{\hat{\nu}}{}^{\hat{\lambda}}{}_{\hat{\mu}} - \hat{Q}^{\hat{\lambda}}{}_{\hat{\mu}\hat{\nu}} \tag{2.3}$$

For the sake of convenience we introduce a modified torsion tensor

$$\hat{T}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} \equiv \hat{Q}_{\hat{\mu}\hat{\nu}}{}^{\hat{\lambda}} + \delta^{\hat{\lambda}}_{\hat{\nu}}\hat{Q}_{\hat{\mu}} - \delta^{\hat{\lambda}}_{\hat{\mu}}\hat{Q}_{\hat{\nu}} \tag{2.4}$$

where $\hat{Q}_{\hat{\mu}} = \hat{Q}_{\hat{\lambda}\hat{\mu}}{}^{\hat{\lambda}}$. The higher dimensional Riemann–Cartan space curvature tensor in coordinate basis is given by

$$\hat{R}_{\hat{\mu}\hat{\nu}\hat{\beta}}{}^{\hat{\lambda}} = 2\partial_{[\hat{\mu}}\hat{\Gamma}^{\hat{\lambda}}_{\hat{\nu}]\hat{\beta}} + 2\hat{\Gamma}^{\hat{\lambda}}_{[\hat{\mu}|\hat{\alpha}}\hat{\Gamma}^{\hat{\alpha}}_{|\hat{\nu}]\hat{\beta}} \tag{2.5}$$

The scalar curvature $\hat{R} = \hat{R}_{\hat{\mu}}{}^{\hat{\mu}} = \hat{R}_{\hat{\mu}\hat{\nu}}{}^{\hat{\nu}\hat{\mu}}$ may be obtained by contraction of the Riemann–Cartan curvature tensor. Using (2.2) and (2.5), we can separate the scalar curvature into a Riemannian and a torsion part,

$$\hat{R} = \hat{\bar{R}} + 2(\hat{\bar{\nabla}}_{\hat{\mu}}\hat{K}_{\hat{\nu}}{}^{\hat{\nu}\hat{\mu}}) + \hat{K}_{\hat{\mu}\hat{\lambda}}{}^{\hat{\mu}}\hat{K}_{\hat{\nu}}{}^{\hat{\nu}\hat{\lambda}} - \hat{K}_{\hat{\nu}\hat{\lambda}}{}^{\hat{\mu}}\hat{K}_{\hat{\mu}}{}^{\hat{\nu}\hat{\lambda}} \tag{2.6}$$

The field Lagrangian density is given by

$$\hat{\mathcal{L}}_G = \frac{1}{2k}\hat{R} \tag{2.7}$$

where $k = 8\pi G$, and G denotes the gravitational constant. Here we use units such that $\hbar = c = 1$.

To introduce a spinor in Riemann–Cartan space, it is necessary to choose at every point of U_n an orthonormal basis. The corresponding n-bein $e^{\hat{M}}{}_{\hat{\mu}}$ satisfies

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}^{\hat{M}}{}_{\hat{\mu}}\hat{e}^{\hat{N}}{}_{\hat{\nu}}\hat{\eta}_{\hat{M}\hat{N}}, \qquad \hat{M}, \hat{N} \cdots = (0, 1, 2, 3, 5, \ldots, n) \tag{2.8}$$

where $\hat{\eta}_{\hat{M}\hat{N}} = \text{diag}(+1, -1, -1, \ldots, -1)$ is the nD Lorentz metric. Just as in the Riemannian case, the spin connection may be written as

$$\hat{\Gamma}_{\hat{\mu}} = \frac{1}{2}\hat{e}_{\hat{M}}{}^{\hat{\nu}}\hat{\nabla}_{\hat{\mu}}(\hat{e}_{\hat{N}\hat{\nu}})\hat{\sigma}^{\hat{M}\hat{N}} \tag{2.9}$$

with $\hat{\sigma}^{\hat{M}\hat{N}} = \frac{1}{4}[\hat{\gamma}^{\hat{M}}, \hat{\gamma}^{\hat{N}}]$, where $\hat{\gamma}^{\hat{M}}$ is the Dirac matrix in nD Lorentz space which satisfies

$$\hat{\gamma}^{\hat{M}}\hat{\gamma}^{\hat{N}} + \hat{\gamma}^{\hat{N}}\hat{\gamma}^{\hat{M}} = 2\hat{\eta}^{\hat{M}\hat{N}} \tag{2.10}$$

Using (2.2), we can rewrite (2.9) as

$$\hat{\Gamma}_{\hat{\mu}} = \hat{\bar{\Gamma}}_{\hat{\mu}} + \hat{K}_{\hat{\mu}} \tag{2.11}$$

where $\hat{K}_{\hat{\mu}} = \frac{1}{2}\hat{K}_{\hat{\mu}\hat{M}\hat{N}}\hat{\sigma}^{\hat{M}\hat{N}}$. The spin covariant derivative, similar to the case of Riemann space, is defined by

$$\hat{\nabla}_{\hat{\mu}}\hat{\Psi} = (\partial_{\hat{\mu}} + \hat{\Gamma}_{\hat{\mu}})\hat{\Psi} = \hat{\bar{\nabla}}_{\hat{\mu}}\hat{\Psi} + \hat{K}_{\hat{\mu}}\hat{\Psi} \tag{2.12}$$

where $\hat{\Psi}$ is a Dirac spinor in higher dimensional space. Let the matter Lagrangian density be given by

$$\hat{\mathcal{L}}_m = \frac{1}{2}i[\hat{\bar{\Psi}}\hat{\gamma}^{\hat{\mu}}\hat{\nabla}_{\hat{\mu}}\hat{\Psi} - \hat{\nabla}_{\hat{\mu}}\hat{\bar{\Psi}}\hat{\gamma}^{\hat{\mu}}\hat{\Psi}] - m\hat{\bar{\Psi}}\hat{\Psi} \tag{2.13}$$

Using (2.11) and the formula

$$\hat{\gamma}^{\hat{\mu}}\hat{\gamma}^{\hat{\nu}}\hat{\gamma}^{\hat{\sigma}} = \hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\sigma}]} + \hat{g}^{\hat{\mu}\hat{\nu}}\hat{\gamma}^{\hat{\sigma}} + \hat{g}^{\hat{\nu}\hat{\sigma}}\hat{\gamma}^{\hat{\mu}} - \hat{g}^{\hat{\sigma}\hat{\mu}}\hat{\gamma}^{\hat{\nu}} \tag{2.14}$$

where $\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\sigma}]} \equiv \hat{\gamma}^{[\hat{\mu}}\hat{\gamma}^{\hat{\nu}}\hat{\gamma}^{\hat{\sigma}]}$ is totally antisymmetric with respect to μ, ν, σ, we can rewrite (2.13) in the form

$$\hat{\mathcal{L}}_m = \hat{\bar{\mathcal{L}}}_m + \frac{1}{4} i\hat{K}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{\bar{\Psi}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\hat{\Psi} \tag{2.15}$$

where $\hat{\bar{\mathcal{L}}}_m$ is the matter Lagrangian density in Riemann space. The total action then is

$$\hat{I} = \int \sqrt{\hat{g}}(\hat{\mathcal{L}}_G + \hat{\mathcal{L}}_m)\, d^n x \tag{2.16}$$

3. FIELD EQUATIONS

In the course of constructing KK theory in higher dimensional Riemann or Weitzenböck space, we can directly reduce the action to a sum of actions of 4D gravity and the gauge field, for the curvature or torsion is formed by the given metric, while in higher dimensional EC theory, both the metric and torsion are independent field variables. In this case, we must first vary the action to obtain the field equations, then consider the possibility of their reduction. Performing the variation with respect to $\hat{g}^{\hat{\mu}\hat{\nu}}$, $\hat{K}_{\hat{\mu}\hat{\nu}\hat{\lambda}}$, and $\hat{\Psi}$, we get the following results:

(i) The first field equation is

$$\hat{\bar{G}}_{\hat{\mu}\hat{\nu}} + \hat{t}_{\hat{\mu}\hat{\nu}} = k\hat{\mathcal{T}}_{\hat{\mu}\hat{\nu}} \tag{3.1}$$

where

$$\begin{aligned} \hat{t}_{\hat{\mu}\hat{\nu}} &= 2[\hat{T}_{\hat{\mu}\hat{\alpha}\hat{\beta}}(\hat{T}_{\hat{\nu}}{}^{\hat{\beta}\hat{\alpha}} + \hat{T}_{\hat{\nu}}{}^{\hat{\alpha}\hat{\beta}}) - \hat{T}_{\hat{\mu}}\hat{T}_{\hat{\nu}}] - 2\hat{T}_{\hat{\alpha}\hat{\beta}\hat{\mu}}\hat{T}^{\hat{\alpha}\hat{\beta}}{}_{\hat{\nu}} \\ &\quad - \frac{1}{2}\hat{g}_{\hat{\mu}\hat{\nu}}[\hat{T}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}(2\hat{T}^{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \hat{T}^{\hat{\alpha}\hat{\beta}\hat{\gamma}}) - 2\hat{T}_{\hat{\alpha}}\hat{T}^{\hat{\alpha}}] \end{aligned} \tag{3.2}$$

labels the combination of the torsion terms with $\hat{T}^{\hat{\mu}} = \hat{T}^{\hat{\lambda}\hat{\mu}}{}_{\hat{\lambda}}$, and

$$\hat{\mathcal{T}}_{\hat{\mu}\hat{\nu}} = \frac{2\delta(\sqrt{\hat{g}}\hat{\bar{\mathcal{L}}}_m)}{\sqrt{\hat{g}}\delta\hat{g}^{\hat{\mu}\hat{\nu}}} = -\frac{i}{2}[\hat{\bar{\Psi}}\hat{\gamma}_{(\hat{\nu}}\hat{\nabla}_{\hat{\mu})}\hat{\Psi} - \hat{\nabla}_{(\hat{\mu}}\hat{\bar{\Psi}}\hat{\gamma}_{\hat{\nu})}\hat{\Psi}] \tag{3.3}$$

denotes the metric energy-momentum tensor. $\hat{\bar{G}}_{\hat{\mu}\hat{\nu}}$ is Einstein tensor in Riemann space. Alternatively, we may introduce an asymmetric total energy-momentum tensor, so as to get the generalized Einstein equation in Riemann–Cartan space. However, it is easy to verify that the two kinds of equation are equivalent.

(ii) The second field equation is

$$\hat{T}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = -k\hat{S}^{\hat{\mu}\hat{\nu}\hat{\lambda}} \tag{3.4}$$

where

$$\hat{S}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = \frac{\delta(\sqrt{\hat{g}}\hat{\mathcal{L}}_m)}{\sqrt{\hat{g}}\delta\hat{K}_{\hat{\mu}\hat{\nu}\hat{\lambda}}} = \frac{1}{4}\, ik\hat{\bar{\Psi}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\hat{\Psi} \tag{3.5}$$

is the spin angular momentum tensor of the matter field. It is totally antisymmetric. From (2.3) and (2.4) we know that $\hat{K}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = -\hat{T}^{\hat{\mu}\hat{\nu}\hat{\lambda}}$ is also totally antisymmetric. Exploiting the second field equation (3.4), we can reduce the first field equation (3.1) to

$$\hat{\tilde{G}}_{\hat{\mu}\hat{\nu}} - \hat{T}_{\hat{\alpha}\hat{\beta}\hat{\mu}}\hat{T}^{\hat{\alpha}\hat{\beta}}{}_{\hat{\nu}} - \frac{1}{2}\,\hat{g}_{\hat{\mu}\hat{\nu}}\hat{T}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}\hat{T}^{\hat{\alpha}\hat{\beta}\hat{\gamma}} = -k\hat{\mathcal{T}}_{\hat{\mu}\hat{\nu}} \tag{3.6}$$

(iii) The Dirac field equation is

$$i\hat{\gamma}^{\hat{\mu}}\hat{\nabla}_{\hat{\mu}}\hat{\Psi} + \frac{1}{4}\, i\hat{K}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\hat{\Psi} - m\hat{\Psi} = 0 \tag{3.7}$$

4. FIVE-DIMENSIONAL EC THEORY

In order to clarify how field equations in higher dimensional EC theory reduce to those of gravity and gauge fields in four-dimensional (4D) spacetime and to reveal the spin–spin interaction induced by torsion explicitly, we consider the simplest 5D case. The 5D metric in coordinate basis may be taken as

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 A_\mu A_\nu & -\kappa A_\mu \\ -\kappa A_\nu & -1 \end{pmatrix}, \qquad \hat{\mu} = (\mu, 5) \tag{4.1}$$

where A_μ may be identified with the electromagnetic potential and the constant κ is introduced to ensure that κA_μ is dimensionless. We choose the horizontal lift basis (HLB) $\hat{\theta}^{\hat{A}}$ with $\hat{A} = (A, 5)$, which is convenient for calculation. Its components are given by

$$\hat{\theta}^A = \delta^A_\mu dx^\mu, \qquad \hat{\theta}^5 = dx^5 + \kappa A_\mu dx^\mu \tag{4.2}$$

The basis vectors $\hat{e}_{\hat{A}}$ which are dual to $\hat{\theta}^{\hat{A}}$ are

$$\hat{e}_A = (\partial_\mu - \kappa A_\mu \partial_5)\delta^\mu_A, \qquad \hat{e}_5 = \partial_5 \tag{4.3}$$

By this choice, the metric becomes diagonal

$$\hat{g}_{\hat{A}\hat{B}} = \begin{pmatrix} g_{AB} & 0 \\ 0 & -1 \end{pmatrix} \tag{4.4}$$

where $g_{AB} = g_{\mu\nu}\delta^{\mu}_{A}\delta^{\nu}_{B}$.

Because the HLB is anholonomic, we define the commutation coefficients $\hat{C}_{\hat{A}\hat{B}}{}^{\hat{E}}$ by

$$[\hat{e}_{\hat{A}}, \hat{e}_{\hat{B}}] = \hat{C}_{\hat{A}\hat{B}}{}^{\hat{E}}\hat{e}_{\hat{E}} \tag{4.5}$$

Just as in coordinate basis, the connection coefficients in HLB can be expressed as

$$\hat{\Gamma}_{\hat{A}\hat{B}\hat{C}} = \hat{\tilde{\Gamma}}_{\hat{A}\hat{B}\hat{C}} - \hat{K}_{\hat{B}\hat{C}\hat{A}}, \tag{4.6}$$

with the Riemann connection coefficients (Toms, 1984)

$$\hat{\tilde{\Gamma}}_{\hat{A}\hat{B}\hat{C}} = \frac{1}{2}\,[\hat{e}_{\hat{E}}(\hat{g}_{\hat{A}\hat{B}}) + \hat{e}_{\hat{B}}(\hat{g}_{\hat{A}\hat{E}}) - \hat{e}_{\hat{A}}(\hat{g}_{\hat{E}\hat{B}}) + \hat{C}_{\hat{A}\hat{B}\hat{E}} + \hat{C}_{\hat{A}\hat{E}\hat{B}} + \hat{C}_{\hat{E}\hat{B}\hat{A}}] \tag{4.7}$$

The curvature tensor in HLB is defined as (Choquet-Bruhat *et al.*, 1977)

$$\hat{R}_{\hat{A}\hat{B}\hat{E}}{}^{\hat{D}} = \hat{e}_{\hat{A}}(\hat{\Gamma}^{\hat{D}}_{\hat{B}\hat{E}}) - \hat{e}_{\hat{B}}(\hat{\Gamma}^{\hat{D}}_{\hat{A}\hat{E}}) + \hat{\Gamma}^{\hat{D}}_{\hat{A}\hat{F}}\hat{\Gamma}^{\hat{F}}_{\hat{B}\hat{E}} - \hat{\Gamma}^{\hat{D}}_{\hat{B}\hat{F}}\hat{\Gamma}^{\hat{F}}_{\hat{A}\hat{E}} - \hat{C}_{\hat{A}\hat{B}}{}^{\hat{F}}\hat{\Gamma}^{\hat{D}}_{\hat{F}\hat{E}} \tag{4.8}$$

Obviously, expression (2.6) also holds in HLB:

$$\hat{R} = \hat{\tilde{R}} + 2(\hat{\tilde{\nabla}}_{\hat{A}}\hat{K}_{\hat{B}}{}^{\hat{B}\hat{A}}) + \hat{K}_{\hat{A}\hat{D}}{}^{\hat{A}}\hat{K}_{\hat{B}}{}^{\hat{B}\hat{D}} - \hat{K}_{\hat{B}\hat{D}}{}^{\hat{A}}\hat{K}_{\hat{A}}{}^{\hat{B}\hat{D}} \tag{4.9}$$

Using HLB, it is easy to prove $\hat{\tilde{R}} = \tilde{R} + \frac{1}{4}\kappa^2F^{\mu\nu}F_{\mu\nu}$; then the field Lagrangian density becomes (in coordinate basis)

$$\hat{\mathcal{L}}_G = \frac{1}{2k}\hat{R} = \frac{1}{2k}\left[\tilde{R} + \frac{1}{4}\kappa^2F^{\mu\nu}F_{\mu\nu} + 2\hat{\tilde{\nabla}}_{\hat{\mu}}\hat{K}_{\hat{\nu}}{}^{\hat{\nu}\hat{\mu}} + \hat{K}_{\hat{\mu}\hat{\lambda}}{}^{\hat{\mu}}\hat{K}_{\hat{\nu}}{}^{\hat{\nu}\hat{\lambda}} - \hat{K}_{\hat{\nu}\hat{\lambda}}{}^{\hat{\mu}}\hat{K}_{\hat{\mu}}{}^{\hat{\nu}\hat{\lambda}}\right] \tag{4.10}$$

Here, we assume $\kappa^2 = 2k = 16\pi G$, so as to make the second term on the right-hand side of (4.10) coincide with the Lagrangian density of electromagnetic field.

In HLB the fünfbein, like the metric, is simply block diagonal

$$e^{\hat{M}}{}_{\hat{A}} = \begin{pmatrix} e^M{}_A & 0 \\ 0 & 1 \end{pmatrix} \tag{4.11}$$

The spin connection in HLB is given by

$$\hat{\Gamma}_{\hat{A}} = \hat{\tilde{\Gamma}}_{\hat{A}} + \hat{K}_{\hat{A}} \tag{4.12}$$

where $\hat{\tilde{\Gamma}}_{\hat{A}} = \frac{1}{2}\hat{e}_{\hat{M}}{}^{\hat{B}}\hat{\tilde{\nabla}}_{\hat{A}}(\hat{e}_{\hat{N}\hat{B}})\hat{\sigma}^{\hat{M}\hat{N}}$ is the Riemann spin connection; its components can be evaluated as (Macias and Dehnen, 1991)

$$\hat{\Gamma}_A = \left(\tilde{\Gamma}_\mu + \frac{1}{4}\kappa e_N{}^\nu F_{\mu\nu}\sigma^{N5}\right)\delta^\mu_A, \qquad \hat{\Gamma}_5 = \frac{1}{4}\kappa e_M{}^\mu e_N{}^\nu F_{\mu\nu}\sigma^{MN} \quad (4.13)$$

The spin-covariant derivatives in HLB are then defined by

$$\hat{\nabla}_{\hat{A}}\hat{\Psi} = (\hat{e}_{\hat{A}} + \hat{\Gamma}_{\hat{A}})\hat{\Psi} \quad (4.14)$$

while the Riemann spin-covariant derivatives are given by

$$\hat{\tilde{\nabla}}_A\hat{\Psi} = \delta^\mu_A(\partial_\mu - \kappa A_\mu\partial_5 + \hat{\tilde{\Gamma}}_\mu)\hat{\Psi}, \qquad \hat{\tilde{\nabla}}_5\hat{\Psi} = (\partial_5 + \hat{\tilde{\Gamma}}_5)\hat{\Psi} \quad (4.15)$$

Therefore, the Dirac Lagrangian density may be taken as

$$\hat{\mathcal{L}}_m = \hat{\tilde{\mathcal{L}}}_m + \frac{1}{4} i\hat{K}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{\bar{\Psi}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\hat{\Psi} \quad (4.16)$$

where

$$\hat{\tilde{\mathcal{L}}}_m = \frac{1}{2} i[\hat{\bar{\Psi}}\hat{\gamma}^{\hat{\mu}}\hat{\tilde{\nabla}}_{\hat{\mu}}\hat{\Psi} - \hat{\tilde{\nabla}}_{\hat{\mu}}\hat{\bar{\Psi}}\hat{\gamma}^{\hat{\mu}}\hat{\Psi}] - m\hat{\bar{\Psi}}\hat{\Psi} \quad (4.17)$$

is the Dirac Lagrangian density in Riemann space. Hence the total action function of 5D EC theory is

$$\hat{I}^5 = \int d^5x\, \sqrt{\hat{g}^5}(\hat{\mathcal{L}}_G + \hat{\mathcal{L}}_m) \quad (4.18)$$

Performing the variation with respect to $\hat{K}_{\hat{\lambda}\hat{\nu}}{}^{\hat{\mu}}$, we obtain the field equation

$$\hat{T}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = -\frac{1}{4} ik\hat{\bar{\Psi}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\hat{\Psi} \quad (4.19)$$

Using (2.3) and (2.4), we have

$$\hat{K}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = -\hat{T}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = \frac{1}{4} ik\hat{\bar{\Psi}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\hat{\Psi} \quad (4.20)$$

Notice that now both the modified torsion tensor $\hat{T}^{\hat{\mu}\hat{\nu}\hat{\lambda}}$ and the contortion tensor $\hat{K}^{\hat{\mu}\hat{\nu}\hat{\lambda}}$ are totally antisymmetric.

Because the fifth coordinate is usually assumed periodic, we choose, as in 5D Riemann space, the Dirac spinor $\hat{\Psi}(x^\mu, x^5) = \exp(ix^5/L)\psi$, to guarantee that the EC theory is covariant with respect to $U(1)$. Thus (4.20) may be expressed by means of a spinor ψ in 4D space-time:

$$\hat{K}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = -\hat{T}^{\hat{\mu}\hat{\nu}\hat{\lambda}} = \frac{1}{4} ik\bar{\psi}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\psi \quad (4.21)$$

To derive the Einstein-like equation of 5D EC theory, we may take the orthonormal fünfbein $\hat{e}^{\hat{M}}{}_{\hat{\mu}}$ as basic field variables, which can be expressed explicitly as

$$\hat{e}^{\hat{M}}{}_{\hat{\mu}} = \begin{pmatrix} e^{M}{}_{\mu} & 0 \\ \kappa A_{\mu} & 1 \end{pmatrix} \tag{4.22}$$

Varying $\hat{I}^5$ with respect to $e^M{}_\mu$ and A_μ, we get the following results:
The Einstein-like equation:

$$\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\tilde{R} = 8\pi G\left[-\frac{1}{2} i(\overline{\psi}\gamma_{(\nu}\hat{\tilde{\nabla}}_{\mu)}\psi - \hat{\tilde{\nabla}}_{(\mu}\overline{\psi}\gamma_{\nu)}\psi) - \left(F^{\lambda}{}_{\mu}F_{\lambda\nu} - \frac{1}{4} g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) + \frac{1}{4} i(16\pi G)^{1/2}F_{\alpha(\mu}\overline{\psi}\sigma_{\nu)}{}^{\alpha}\psi \right] + \frac{1}{16}\kappa^2\overline{\psi}\hat{\gamma}_{[\mu\hat{\alpha}\hat{\beta}]}\psi\overline{\psi}\hat{\gamma}_{[\nu}{}^{\hat{\alpha}\hat{\beta}]}\psi - \frac{1}{32} g_{\mu\nu}\kappa^2\overline{\psi}\hat{\gamma}_{[\hat{\alpha}\hat{\beta}\hat{\lambda}]}\psi\overline{\psi}\hat{\gamma}^{[\hat{\alpha}\hat{\beta}\hat{\lambda}]}\psi \tag{4.23}$$

The Maxwell equation:

$$\tilde{\nabla}_{\nu}F^{\mu\nu} = -e\overline{\psi}\gamma^{\mu}\psi - \tilde{\nabla}_{\nu}[\tfrac{1}{2}i(16\pi G)^{1/2}\overline{\psi}\sigma^{\mu\nu}\gamma^5\psi] \tag{4.24}$$

We can also vary $\hat{I}^5$ with respect to ψ, and obtain the following:
The Dirac equation:

$$i\hat{\gamma}^{\hat{\mu}}\hat{\tilde{\nabla}}_{\hat{\mu}}\psi + \frac{1}{4} i\hat{K}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{\gamma}^{[\hat{\mu}\hat{\nu}\hat{\lambda}]}\psi - m\psi = 0 \tag{4.25}$$

Using (4.21), we can rewrite (4.25) as

$$i\gamma^{\mu}(\partial_{\mu} - ieA_{\mu} + \tilde{\Gamma}_{\mu})\psi + \frac{1}{2} i(16\pi G)^{1/2}F_{\mu\nu}\sigma^{\mu\nu}\gamma^5\psi - \frac{1}{L}\gamma^5\psi - m\psi - 3\pi G(\overline{\psi}\gamma_5\gamma^{\mu}\psi)\gamma_5\gamma_{\mu}\psi - \frac{3}{2}\pi G(\overline{\psi}\gamma_{[5\mu\nu]}\psi)\gamma^{[5\mu\nu]}\psi = 0 \tag{4.26}$$

where $e = \kappa/L$ is the electron's charge.

It can be seen that, in contrast with the Einstein equation in 5D Riemann space (Macias and Dehnen, 1991), the Einstein-like equation (4.23) has two extra terms. The first of them exists in the 4D EC theory already, while the last one is specific to the 5D case. Both of them may be regarded as additional effective energy-momentum tensors arising from spin–spin interaction induced by torsion. The analogous terms also appear in the Dirac equation (4.26). But the Maxwell equation (4.24) has the same form as in the Riemannian case. This reflects the fact that there is no coupling between torsion and electromagnetic fields.

Just as in the case of 5D Riemann space, there is an additional mass term $L^{-1}\gamma^5\psi$ in the Dirac equation (4.26) of 5D Riemann–Cartan space. The large magnitude of the additional term $[L^{-1} = e/(16\pi G)^{1/2} \simeq 2.6 \times 10^{-7}$ g] yields no correspondence between Dirac fields and observed particles. There are two possible approaches to obviate this problem. First, we may perform a spinor transformation (Vladimirov, 1987), $\psi = S\psi' = \exp(\theta\gamma^5)\psi'$, to diagonalize the mass term by appropriate choice of θ and get the total mass $m' = (L^{-2} + m^2)^{1/2}$, which consists of two parts: the mass contribution from the electrical charge and the prime mass m. In order to match the m' with the observed electron mass, for example, the prime mass m must be taken to be imaginary. Macias and Dehnen (1992) have proposed another approach to solving the mass problem by introducing a scalar field into the metric of the Kaluza–Klein theory of 5D Riemann space. In this approach the coupling term of scalar and Dirac fields can be interpreted as the mass term, and a nontrivial ground state for the scalar field exists. Then the observed electron mass can be obtained by appropriate choice of the ground state of the scalar field. It can be easily seen that the second approach should also be applicable for the 5D Riemann–Cartan case.

5. THE SPIN–SPIN INTERACTION IN 5D EC THEORY

From the field equation (4.20), we can see that the term which contains the contortion tensor in the Dirac equation (4.25) represents the spin–spin interaction induced by torsion. To single out this interaction, we may ignore the Riemann connection ($\hat{\Gamma}_\mu$) and electromagnetic potential (A_μ) temporarily. Thus the Dirac equation (4.25) can be simplified as

$$i\hat{\gamma}^{\hat{\mu}}\partial_{\hat{\mu}}\psi + \frac{1}{4}\, i\hat{K}_{[\hat{\alpha}\hat{\beta}\hat{\gamma}]}\hat{\gamma}^{[\hat{\alpha}\hat{\beta}\hat{\gamma}]}\psi - m\psi = 0 \qquad (5.1)$$

According to the discussion at the end of the last section, here m should be regarded as the total mass m'.

For simplicity, suppose that the contortion tensor $\hat{K}_{[\hat{\alpha}\hat{\beta}\hat{\gamma}]}$ can be regarded as the background torsion generated by the spin-$\frac{1}{2}$ particles, which is proposed as an electron distribution of number density n and spin in the up $(+z)$ direction. The background electron wave function in terms of which the $\hat{K}_{[\hat{\alpha}\hat{\beta}\hat{\gamma}]}$ is formed may be taken as $u = u(0)e^{ip_0 t}$, $u(0) = \sqrt{n}(1\ 0\ 0\ 0)^T$. Put a test electron at rest in the background; it will suffer an action from the background torsion. Its wave function can be set as $\psi = \psi(0)e^{iEt}$. Then the Dirac equation becomes

$$\gamma^0 m\psi + (3\pi G)(\bar{u}\gamma_5\gamma^\alpha u)\gamma_5\gamma_\alpha\psi + \frac{3}{2}\pi G(\bar{u}\gamma_{[5\alpha\beta]}u)\gamma^{[5\alpha\beta]}\psi = E\psi \quad (5.2)$$

In the aforementioned case, it is reasonable to use the constant Dirac matrices

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma_5 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (5.3)$$

satisfying (2.10). Then we get from (5.2) the eigenenergy for the test particle (in cgs units)

$$E = mc^2 \pm \left(\frac{3\pi G\hbar^2}{c^2} - \frac{\pi G\hbar^2}{3c^2}\right)n = mc^2 \pm \frac{8\pi G\hbar^2 n}{3c^2} \quad (5.4)$$

Notice that in (5.2), the nonlinear term $(3\pi G)(\bar{u}\gamma_5\gamma^\alpha u)\gamma_5\gamma_\alpha\psi$ which exists in 4D EC theory already stands for the spin–spin pseudovector direct interaction term (Hehl and Datta, 1971), while the second term $\frac{3}{2}\pi G(\bar{u}\gamma_{[5\alpha\beta]}u)\cdot\gamma^{[5\alpha\beta]}\psi$, induced due to the additional dimension of space, represents the spin–spin pseudotensor direct interaction term, which has an opposed effect on energy to that of the pseudovector one.

On the right-hand side of (5.4), the positive sign corresponds to test particle spin in the up direction, aligned with the background, and the negative sign to test particle spin opposed to the background. The same result is obtained for positrons. Thus, we may conclude that the gravitational spin–spin interaction is repulsive for Dirac particles with aligned spins and attractive for opposed spins. Because of the extremely small order of magnitude ($8\pi G\hbar^2/3c^2 \sim 10^{-82}$ erg·cm^3), the spin–spin interaction can be comparable to the mass term whenever the number density $n = m/k^2\hbar = mc^4/8\pi G\hbar \sim 10^{75}$ cm^{-3} is achieved; therefore, it might have an effect only on the evolution of the early universe (Kerlick, 1975; Hehl *et al.*, 1976).

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REFERENCES

Choquet-Bruhat, Y., DeWitt-Morette, C., and Dillard-Bleick, M. (1977). *Analysis, Manifolds and Physics*, North-Holland, Amsterdam.

Hayashi, K., and Shirafuji, T. (1979). *Physical Review D*, **19**, 3524.
Hehl, F. W., and Datta, B. K. (1971). *Journal of Mathematical Physics*, **12**, 1334.
Hehl, F. A., von der Heyde, P., and Kerlick, G. D. (1976). *Reviews of Modern Physics*, **48**, 393.
Kaluza, Th. (1921). *Sitzungsberichte der K. Preussischen Akademie der Wissenschaften zu Berlin*, p. 966.
Kerlick, G. D. (1975). *Physical Review D*, **12**, 3004.
Klein, O. (1926). *Zeitschrift für Physik*, **37**, 895.
Macias, A., and Dehnen, H. (1991). *Classical and Quantum Gravity*, **8**, 203.
Macias, A., and Dehnen, H. (1992). *International Journal of Modern Physics A*, **7**, 5015.
Toms, D. J. (1984). In *An Introduction to Kaluza–Klein Theories*, H. C. Lee, ed., World Scientific, Singapore, p. 185.
Vladimirov, Yu. S. (1987). *Dimension of Physical Space-Time and Unification of Interactions*, *Moscow University Press, Moscow* [*in russian*].
Youlin, W. (1993). Scientia Sinica, **36A**, 210.